Hydrodynamics in Highly Singular Potentials

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Abstract: There exists a family of radial potentials that lives on the boundary between those well behaved and those highly singular. I have investigated the vortex structures formed in an inverse square potential in the context of weakly interacting classical particles, classical dilute fluids in the low temperature limit, degenerate quantum gases (BECs) and ultracold non-degenerate quantum gases. It can be shown that the only vorticity permitted in the system is at the singularity and that any angular momentum present in the system must take the form of irrotational quantum vortices about the origin. Capture cross sections are found and an experiment is suggested to demonstrate the existence of high-index quantum vortices.

INTRODUCTION

Potentials of the form r^{-n} have been actively studied for more than 50 years. There exists a strict condition for the inclusion of bound states of n<2 which led physicists of the early 20th century to discount more singular potentials as "unphysical". Mathematicians however are rarely deterred by such trivialities and continued to develop a classical and quantum mechanical framework that allowed them to work with systems that contained no intrinsic length or energy scales. In this paper I will describe the behavior of an ideal, dilute bosonic superfluid in a potential of the form n=2 as a system of concentric, rapidly collapsing quantum vortices closely related to coreless Rankine vortices. Continuum Mechanics, however, is a very fluid subject, so although one may be tempted to just go with the flow and dive right in, I feel it best that we first get our feet wet with a little classical mechanics. Therefore, we will begin by describing the behavior of a single particle in the framework of Hamiltonian mechanics. These results will then give us a general form for the velocity profile we will seek while solving the classical Navier-Stokes equations. Finally we will solve the full quantum mechanical system by taking the appropriate approximations of the Gross-Piteavskii equation for weakly interacting bosons.

CLASSICAL DESCRIPTION

For a generic potential U[r] The Legrangian for the system in cylindrical coordinates takes the form:

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + U$$

From this we find three generalized momenta:

$$P_r = m\dot{r} \qquad P_z = m\dot{z} \qquad L = mr^2\dot{\phi}$$

Since U only depends on r the generalized momenta corresponding to z and ϕ are conserved and we can associate them with linear and angular momentum respectively.

By taking a Legendre transformation of $\mathcal{L}$ and setting $P_z = 0$ we arrive at the effective 1D Hamiltonian:

$$\mathcal{H} = \frac{P_r^2}{2m} + \frac{L^2}{2mr^2} + U[r] = \frac{P_r^2}{2m} + U_{eff}[r]$$

We are now in a position to get a sense of how matter will behave in radial potentials of the form $-1/r^n$. For n<2, the repulsive centrifugal term will dominate at small r while the attractive interaction term will dominate at large r creating stable bound states. For n>2 this is reversed, far away the particle is repelled by the centrifugal term and while close to the origin it is strongly attracted creating a potential barrier with an unstable equilibrium. If you find yourself curious about these unstable equilibriums, look up bubble nucleation

The case for n=2 is a very special one. For this we let $U[r] = -K^2/2mr^2$ which leads us to the effective Hamiltonian:

$$\mathcal{H} = \frac{P_r^2}{2m} + \frac{L^2 - K^2}{2mr^2}$$

This precludes the formation of any equilibrium states (stable or unstable) and is highly dependent on the sign of $L^2 - K^2$. For $L^2 > K^2$ this potential is completely repulsive and V_r will always grow more and more positive while an incoming particle with $L^2 < K^2$ will feel a completely attractive potential drawing it tighter and tighter into the trap without the possibility of either forming a stable bound state or escaping to infinity. Hamilton's equations for this system yield the following coupled PDEs that govern particle's motion:

(i) $\frac{\partial L}{\partial t} = 0$ $\qquad$ $\frac{\partial \phi}{\partial t} = \frac{L}{mr^2}$ (ii)

(iii) $\frac{\partial P_r}{\partial t} = \frac{L^2 - K^2}{mr^3}$ $\qquad$ $\frac{\partial d}{\partial t} = \frac{P_r}{m}$ (iv)

From equation (ii) we see:

$$V_\phi = r\frac{\partial \phi}{\partial t} = \frac{L}{mr}$$

By expanding equation (iii) via the chain rule and combining the result with equation (iv) we arrive at the characteristic ODE describing the radial motion of the particle:

$$\frac{\partial P_r}{\partial t} = \frac{\partial P_r}{\partial r}\frac{\partial r}{\partial t} \quad \Rightarrow \quad \frac{\partial V_r}{\partial r} = \frac{L^2 - K^2}{V_r\, m^2 r^3}$$

Solving this by separation of variables yields:

$$V_r = \pm\sqrt{C_1 - \frac{L^2 - K^2}{m^2 r^2}}$$

Where the + refers to the solution of $L^2 > K^2$ and the – to solutions of $L^2 < K^2$. If a particle is given an initial velocity such that the potential causes the velocity to change sign (e.g. $V_{r,i} = -V_0\hat{r}$ and $L^2 > K^2$) then $V_r[r]$ may be double valued. C_1 is a constant that will account for the boundary and initial conditions

CONTINUUM MECHANICS

In an effort to cast this problem into framework of continuum mechanics, we will search for solutions to the Navier-Stokes equations for a compressible ideal gas in the low temperature limit. Later, we will quantify these requirements further but for now let it suffice to say that we will restrict ourselves to densities and temperatures such that $K_b T \ll \langle U_{eff}\rangle$ and atom-atom interactions are negligible.

The governing equations in this limit are:

(v) $\quad \partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = S[r]$

(vi) $\quad \rho\,\partial_t \vec{v} + \rho\big(\vec{v}\cdot\vec{\nabla}\big)\vec{v} = -\vec{\nabla}P + \vec{f}$

Where S[r] is a position dependent mass source term describing the rate and location that mass is being added to the system and ρ, v and P are the usual position dependent mass density, velocity and pressure respectively.

To proceed we need to find a state function that relates the pressure to other variables in the system such as the ideal gas law $P = \eta K_b T$, where η is the particle density. Now, while this relation does not allow us to directly disregard the pressure term, as both η and $K_b T$ are small, we can directly compare $K_b T$ to U[r] by rewriting $\vec{f}$, which is the body force per unit volume, as:

$$\vec{f} = \eta\vec{F} = -\eta\vec{\nabla}U = -\vec{\nabla}(\eta U) + U\vec{\nabla}\eta$$

where $\vec{F}$ is the force per particle.This form allows us to combine the momentum source term (RHS of equation vi) into:

$$-\vec{\nabla}(\eta K_b T) + \vec{f} = -\nabla[\eta(K_b T + U)] + U\nabla\eta$$

Now, taking the approximation $K_b T \ll \langle U_{eff}\rangle$, the momentum balance equation simplifies to:

$$\rho\,\partial_t\vec{v} + \rho\big(\vec{v}\cdot\vec{\nabla}\big)\vec{v} = -\eta\vec{\nabla}U$$

In the case of the inverse square potential in steady state, accounting for all the proper symmetries, the Navier-Stokes equations reduce to:

(vii) $\quad \partial_r(r\rho v_r) = rS[r]$

(viii) $\quad v_r\partial_r v_r - \frac{V_\phi^2}{r} = -\frac{\eta^2 K^2}{\rho^2 r^3}$

(ix) $\quad \partial_r v_\phi = -\frac{v_\phi}{r}$

Equation ix can be directly solved by separation of variables leading to:

$$V_\phi = \frac{C_2}{r} \equiv \frac{\bar{L}}{r}$$

Where $\bar{L} \equiv L\,\eta/\rho$ is the conserved quantity we can identify with angular momentum per unit mass. Plugging this result into equation viii, we get the continuum equation governing radial flow:

$$\partial_r v_r = \frac{\bar{L}^2 - \bar{K}^2}{v_r\, r^3}$$

where $\bar{K} \equiv K\,\eta/\rho$ is the potential constant per unit mass. We can now write down an explicit form for the radial flow in steady state:

$$V_r = \pm\sqrt{C_1 - \frac{\bar{L}^2 - \bar{K}^2}{r^2}}$$

As in the single particle case, the sign of V_r and the value of the constant C1 are determined by the boundary conditions.

BOUNDARY CONDITIONS

The solutions found above are general and only by applying them to specific situations do we get useful results. One such case would be to place this potential in an infinite, stagnant pool of density ρ_0. This yields the boundary conditions:

$$V_\phi = 0 \qquad V_r[\infty] = 0 \qquad S[r] = 0$$

Plugging these into equation vii and the general velocity profiles found above we arrive at the full solution.

$$V_r = -\frac{\bar{K}}{r} \qquad \rho[r] = \rho_0$$

We see that the density is constant; this is a result of the velocity's dependence on r exactly countering the differential volume element. The streamlines for this system are shown in figure 1. It is interesting to note that, while the solution for $\rho[r]$ is not valid at the origin, it is valid infinitesimally close.

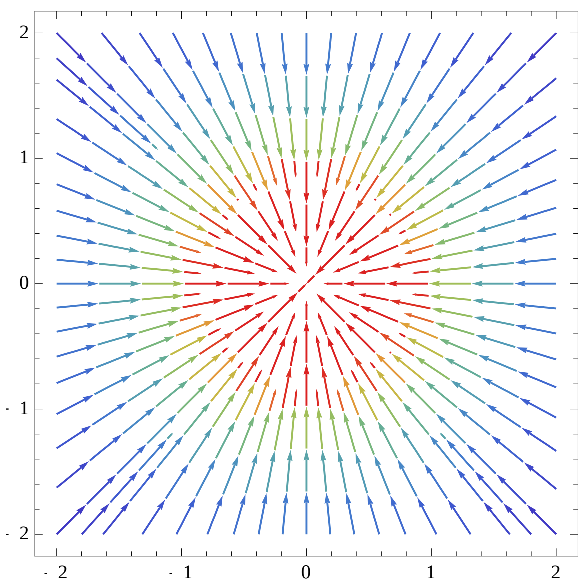

Figure 1: Streamlines from stagnant pool at infinity. Color indicates speed with red signifying greater velocities than blue.

A second, and perhaps more interesting, problem is that of the resultant flow from firing a constant stream of atoms into the system with density, velocity and impact parameter ρ_0, v_0 and r_0 respectively. This can be represented by the inclusion of a non-trivial source term along with proper boundary conditions. Let:

$$\bar{L} = r_0 v_0 \quad V_r[r_0] = 0$$

$$S[r] = \rho_0 v_0 \delta[r - r_0]$$

Integrating equation vii and solving for all constants leads us to the results:

$$V_r = \pm\frac{1}{r}\sqrt{(\bar{L}^2 - \bar{K}^2)(\frac{r^2}{r_0^2} - 1)}$$

$$V_\phi = \frac{r_0 v_0}{r}$$

$$\rho = \frac{\rho_0 v_0 r_0}{\sqrt{(\bar{L}^2 - \bar{K}^2)(\frac{r^2}{r_0^2} - 1)}}$$

As before, the positive solution corresponds to the fields generated by $L^2 > K^2$ and the negative solution corresponds to the fields generated by $L^2 < K^2$. This leaves us with 2 distinct regimes. If incoming matter has an angular momentum about the origin greater than some critical value L_c then real solutions to the Navier-Stokes equations can only be found for $r > r_0$ and the incoming fluid will flow toward infinity. On the other hand, if the incoming fluid has a low enough angular momentum then all real solutions are bounded to the region $r < r_0$ in which case the flow is toward the origin.

We can now imagine the more realistic situation of sending in a parcel of fluid with finite spatial extent. If the parcel has density ρ_0 and is traveling with velocity v_0, there will exist a spread in angular momentum across the incoming fluid due to variation of the impact parameter. Under these conditions the critical angular momentum per unit mass $\bar{L}_c$ can be rewritten as a critical impact parameter $b_c = \bar{L}_c/v_0 = \bar{K}/v_0$. If the entire parcel is located outside of this critical radius then all the fluid will be deflected; if it is located entirely within this radius then it will be completely captured, but if b_c is contained within the spread of incoming impact parameters, then the parcel will be split into two with the two halves following very different trajectories. This leads us to define $2b_c$ to be the effective 1 dimensional capture cross section.

Regardless of which path is taken the vorticity, defined as:

$$\vec{\omega} = \vec{\nabla} \times \vec{v} = 0 \quad \text{for all } r \neq 0$$

The circulation around any path that does not enclose r=0 can be shown to be 0 by applying stokes theorem. If, however, a path is chosen that includes the pole at the origin the circulation can take a non-zero value; more specifically:

$$\Gamma = \oint_0^{2\pi} v_\phi r d\phi = 2\pi \bar{L}$$

Our system can support finite circulation of any real value.

QUANTUM MECHANICS

The quantum nature of an ultra-cold N particle system of weakly interacting bosons is described by the nonlinear Gross-Pitaevskii equation:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U[r] + \frac{4\pi\hbar^2 a_s}{m}|\psi|^2\right)\psi = \mu\,\psi$$

Where the nonlinear term accounts for the weak boson-boson interaction and is proportional to a_s, the boson-boson scattering length.

We can now define more precisely the condition for the system to be "non-interacting", specifically:

$$U[r] \gg \frac{4\pi\hbar^2 a_s}{m}|\psi|^2 = \frac{4\pi\hbar^2 a_s}{m}\eta$$

Using realistic numbers for Rubidium-87 with scattering length on the order of 100 times the Bohr radius and densities of $10^{14}\ cm^{-3}$ we get interaction energies of $10^{-12} eV$.

Under the non-interacting approximation we can cast the Gross-Pitaevskii equation into cylindrical coordinates and set U[r] to be our inverse square potential to get:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{L^2 - K^2}{2m\,r^2}\right)\psi = \mu\,\psi$$

Assume ψ is separable such that it can be written:

$$\psi[r,\phi,z] = R[r]\;\Phi[\phi]\;Z[z]$$

Plugging this into the Gross-Pitaevskii equation we get a free particle in two dimensions and a second order nonlinear ODE in r. Fortunately, the resultant ODE is solvable, leaving us with a general form for the separable wavefunction:

$$Z[z] = e^{ikz} \qquad \Phi[\phi] = e^{il\phi}$$

$$R[r] = C_1\,J_\alpha\left[\sqrt{\epsilon}\,r\right] + C_2\,Y_\alpha\left[\sqrt{\epsilon}\,r\right]$$

$$\alpha \equiv l^2 - \frac{2mK^2}{\hbar^2} \qquad \epsilon \equiv \frac{2m\mu}{\hbar^2},$$

Where J_n and Y_n are Bessel functions of the first and second kind respectively of order n, l and k are quantum numbers and the $C_i's$ are constants to be determined by the boundary conditions.

For the problem being addressed in this paper we need only note that the only ϕ dependence of ψ is of the form $e^{il\phi}$. The probability density current in the azimuthal direction, which can be identified as the quantum mechanical equivalent to fluid velocity, can then be written:

$$j_\phi = -\frac{i\hbar}{2m} < \psi^*\nabla_\phi\psi - \psi\nabla_\phi\psi^* >= \frac{\hbar l}{rm}$$

Where l can be any value that satisfies the periodic boundary condition in ϕ of:

$$\psi[r,\phi,z] = \psi[r,\phi + 2\pi, z]$$

$$\Longrightarrow \qquad l = 0, \pm 1, \pm 2, \ldots$$

This means that at a given location of r, the azimuthal velocity in the system can only assume quantized values. The vorticity, as defined above, is identical to the classical system; that is to say that it is zero everywhere but the origin. The true quantum nature of this system comes into play when we inspect the circulation.

$$\Gamma = \oint_0^{2\pi} v_\phi r d\phi = \left(\frac{h}{m}\right) l$$

Any angular momentum present in the system, must take the form of concentric quantum vortices with circulation quantized in units of the intrinsic circulation h/m, where h is the plank constant.

Similar to the classical system, we can consider the capture cross section of an incoming parcel of fluid for a given potential constant K. The incoming fluid must take discrete values of angular momentum $L_i = l\hbar$ resulting in discrete jumps in the capture cross section.

EXPERIMENTAL VERIFICATION

To verify the existence and directly measure the predicted superfluidic quantum vortices, we must first find a way of generating a cylindrically symmetric, inverse square potential. Originally deemed “unphysical”, as Maxwell’s equations forbid a $1/r^2$ falloff of electric and magnetic fields in free space, the nonlinear dipole interaction gives us a chance. The internal energy of a neutral, polarizable atom in a static electric field is given by:

$$U[r] = -\frac{\alpha E^2}{2}$$

where α is the atomic polarizibility. To create the needed inverse square potential we simply need to find a cylindrically symmetric system that produces a static electric field with radial dependence 1/r. This is precisely the field created by an infinitely thin, infinitely long, uniformly charged wire.

I propose an experiment where a cloud of neutral Rubidium-87 bosons is cooled to a temperature below the critical value required for Bose-Einstein Condensation and launched at a single walled, suspended carbon nanotube charged to sufficiently high voltages. The critical angular momentum equal to the potential constant K is then given by:

$$L_c^2 = K^2 = \frac{\alpha\, m\, V_{NT}^2}{Ln^2\left[\frac{R}{r_0}\right]}$$

where r_0 is the radius of the nanotube and R is the radius of a large effective grounding cylinder.

Atoms with sufficiently low orbital angular momentum will spiral into the wire forming concentric quantized vortices. As they do so, they will be forced into regions of higher and higher electric field resulting in two possible scenarios. If the voltage on the tube is small, then atoms will collide with the wire and be stuck, but if the potential gradient near the tube is larger than the ~3V/nm holding the atom together, then the electron will tunnel into the nanotube, field ionizing the once neutral Rubidium. The resultant ion will be violently ejected and can then be detected at the single ion level by a standard channel-electron multiplier. For a nanotube radius of 3.3nm the critical voltage is ~140V

When operated in the high voltage regime, this setup will enable a direct measure of how many atoms are captured by the wire. As V_{NT} is ramped linearly, discrete quantum vortices should be directly seen as discrete jumps in ion count. This would constitute the first direct measurement of high-index quantum vortices. Previous setups have demonstrated ground state vortices in condensates, but as more angular momentum is added, the system has always been such that it energetically prefers the creation of N ground state vortices.

ADDITIONAL CONSIDERATIONS

Along with the direct observation of quantized circulation, the suspended nanotube can act as a highly sensitive neutral atom detector. Similar devices currently operate by using a macroscopic hot wire to thermally ionize atoms resulting in a spatial resolution that is limited by the size of the wire and surface diffusion. By implementing a position sensitive microchannel plate, along with ion optics, the spatial resolution along the axis of the nanotube can approach the single nanometer level with a temporal resolution in the nanosecond regime at room temperature. This will allow for the development of chip-based, neutral atom interferometers with novel applications in precision navigation and general relativity experiments. The strong dependence on the atom’s polarizability can provide a new measurement of α as well as allow for the creation of new, highly sensitive, chip based gas detectors capable of operating at the single particle level. These detectors have immediate applications in the detection of malicious gases and in the detection of trace gases emitted by cells that are undergoing stress. Interesting experiments have also been proposed in which an AC potential is applied to the nanotube allowing for the creation of bound states of atoms orbiting the nanotube. Multiple atoms could be confined in tight orbits inducing strong dipole-dipole coupling and potentially resulting in new, highly correlated states that can be directly probed by the nanotube detector. Bound states can also be formed through the coupling of the magnetic dipole moment in a neutral particle to the 1/r magnetic potential that arises from running a steady current through the nanotube, an idea originally proposed in 1991 for neutrons. Through an extension of the concept of supersymmetry it can be shown that atoms bound in this potential will display a hydrogenic spectrum.

It is worth noting that this setup is currently being constructed and measurements are expected within a year.